

# THE DIFFERENTIAL STRUCTURE OF THE BRIESKORN LATTICE

MATHIAS SCHULZE

**ABSTRACT.** The Brieskorn lattice  $H''$  of an isolated hypersurface singularity with Milnor number  $\mu$  is a free  $\mathbb{C}\{\{s\}\}$ -module of rank  $\mu$  with a differential operator  $t = s^2\partial_s$ . Based on the mixed Hodge structure on the cohomology of the Milnor fibre, M. Saito constructed  $\mathbb{C}\{\{s\}\}$ -bases of  $H''$  for which the matrix of  $t$  has the form  $A = A_0 + A_1s$ . We describe an algorithm to compute the matrices  $A_0$  and  $A_1$ . They determine the differential structure of the Brieskorn lattice, the spectral pairs and Hodge numbers, and the complex monodromy of the singularity.

## 1. THE MILNOR FIBRATION

Let  $f : (\mathbb{C}^{n+1}, \underline{0}) \longrightarrow (\mathbb{C}, 0)$  be a holomorphic function germ with an isolated critical point and Milnor number  $\mu = \dim_{\mathbb{C}} \mathbb{C}\{\underline{x}\} / \langle \partial(f) \rangle$  where  $\underline{x} = x_0, \dots, x_n$  is a complex coordinate system of  $(\mathbb{C}^{n+1}, 0)$  and  $\underline{\partial} = \partial_{x_0}, \dots, \partial_{x_n}$ . By the finite determinacy theorem, we may assume that  $f \in \mathbb{C}[\underline{x}]$ . By E.J.N Looijenga [7, 2.B], for a good representative  $f : X \longrightarrow T$  where  $T \subset \mathbb{C}$  is an open disk at the origin, the restriction  $f : X' \longrightarrow T'$  to  $T' = T \setminus \{0\}$  and  $X' = X \setminus f^{-1}(0)$  is a  $\mathcal{C}^\infty$  fibre bundle unique up to diffeomorphism, the Milnor fibration. By J. Milnor [9, 6.5], the general fibre  $X_t = f^{-1}(t)$ ,  $t \in T'$ , is homotopy equivalent to a bouquet of  $\mu$   $n$ -spheres and, in particular, its reduced cohomology is  $\tilde{H}^k(X_t) \cong \delta_{k,n} \mathbb{Z}^\mu$  where  $\delta$  is the Kronecker symbol. Since  $T'$  is locally contractible, the  $n$ -th cohomologies  $H(U) = H^n(X_U)$  of  $X_U = f^{-1}(U)$  form a locally free  $\mathbb{Z}$ -sheaf of rank  $\mu$  and  $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$  is a complex local system of dimension  $\mu$ . Hence, the sheaf of holomorphic sections  $\mathcal{H} = H \otimes_{\mathbb{Z}} \mathcal{O}_{T'}$  of  $H_{\mathbb{C}}$  is a locally free  $\mathcal{O}_{T'}$ -sheaf of rank  $\mu$ , the cohomology bundle. By P. Deligne [4, 2.23], there is a natural flat connection  $\nabla : \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathcal{O}_{T'}} \Omega_{T'}^1$  on  $\mathcal{H}$  with sheaf of flat sections  $H = \ker(\nabla)$ , the Gauss-Manin connection.

## 2. THE MONODROMY REPRESENTATION

Let  $t$  be a complex coordinate of  $T \subset \mathbb{C}$ ,  $i : T' \longrightarrow T$  the canonical inclusion, and  $u : T^\infty \longrightarrow T'$  the universal covering of  $T'$  defined by  $u(\tau) = \exp(2\pi i \tau)$  for a complex coordinate  $\tau$  of  $T^\infty \subset \mathbb{C}$ . Then the covariant derivative  $\nabla_{\partial_t}$  of  $\nabla$  along  $\partial_t$  induces a differential operator  $\partial_t$  on  $i_*\mathcal{H}$  and the pullback  $f^\infty : X^\infty = X' \times_{T'} T^\infty \longrightarrow T^\infty$

is a  $\mathcal{C}^\infty$  fibre bundle with  $X_\tau^\infty = X_{u(\tau)}$ , the (canonical) Milnor fibre. Since  $T^\infty$  is contractible, the  $n$ -th cohomologies  $H(U) = H^n(X_U^\infty)$  of  $X_U^\infty = (f^\infty)^{-1}(U)$  form a free  $\mathbb{Z}$ -sheaf of rank  $\mu$  and  $u_*H$  is the sheaf of multivalued sections of  $H$ . Lifting closed paths in  $T'$  along sections of  $H$  defines the monodromy representation  $\pi_1(T', t) \longrightarrow \text{Aut}(H_t)$  on  $H_t$  inducing the monodromy representation  $\pi_1(T') \longrightarrow \text{Aut}(H)$  on the cohomology  $H$  of the Milnor fibre. The image  $M$  of the counterclockwise generator of  $\pi_1(T')$  is called the monodromy operator and fulfills  $M(s)(\tau) = s(\tau + 1)$  for  $s \in H$ . The sheaf  $H$  is determined by the monodromy representation up to isomorphism. The following well known theorem is due to E. Brieskorn [2, 0.6] and others.

**Theorem 1** (Monodromy Theorem). *The eigenvalues of the monodromy are roots of unity and its Jordan blocks have size at most  $(n+1) \times (n+1)$  and size at most  $n \times n$  for eigenvalue 1.*

### 3. THE GAUSS-MANIN CONNECTION

Let  $M = M_s M_u$  be the decomposition of  $M$  into semisimple part  $M_s$  and unipotent part  $M_u$  and let  $N = -\frac{\log M_u}{2\pi i}$  be the nilpotent part of  $M$ . Note that  $-2\pi i N \in \text{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$  where  $H_{\mathbb{Q}} = H \otimes_{\mathbb{Z}} \mathbb{Q}$ . Let  $H_{\mathbb{C}} = \bigoplus_{\lambda} H_{\mathbb{C}}^{\lambda}$  be the decomposition of  $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$  into generalized  $\lambda$ -eigenspaces  $H_{\mathbb{C}}^{\lambda}$  of  $M$  and  $M^{\lambda} = M|_{H_{\mathbb{C}}^{\lambda}}$ . Note that  $H_{\mathbb{Q}} = H_{\mathbb{Q}}^1 \oplus H_{\mathbb{Q}}^{\neq 1}$  where  $H_{\mathbb{Q}}^1 \otimes_{\mathbb{Q}} \mathbb{C} = H_{\mathbb{C}}^1$  and  $H_{\mathbb{Q}}^{\neq 1} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\lambda \neq 1} H_{\mathbb{C}}^{\lambda}$ . Then there is an inclusion

$$H_{\mathbb{C}}^{e^{-2\pi i \alpha}} \xrightarrow{\psi_{\alpha}} (i_* \mathcal{H})_0$$

defined by  $\psi_{\alpha}(A) = t^{\alpha+N} A = t^{\alpha} \exp(N \log(t))$  with image  $C^{\alpha} = \text{im}(\psi_{\alpha})$ . In particular, the operators  $M$  and  $N$  act on  $C^{\alpha}$ . The following lemma is an immediate consequence of the definition of  $\psi_{\alpha}$ .

**Lemma 2.**

- (1)  $t \circ \psi_{\alpha} = \psi_{\alpha+1}$  and  $\partial_t \circ \psi_{\alpha} = \psi_{\alpha-1} \circ (\alpha + N)$ .
- (2)  $t : C^{\alpha} \longrightarrow C^{\alpha+1}$  is bijective and  $\partial_t : C^{\alpha} \longrightarrow C^{\alpha-1}$  is bijective if  $\alpha \neq 0$ .
- (3) On  $C^{\alpha}$ ,  $t\partial_t - \alpha = N$  and  $\exp(-2\pi i t \partial_t) = M^{e^{-2\pi i \alpha}}$ .
- (4)  $C^{\alpha} = \ker(t\partial_t - \alpha)^{n+1}$ .

**Definition 3.** We call  $G = \bigoplus_{-1 < \alpha \leq 0} \mathbb{C}\{t\}[t^{-1}]C^{\alpha} \subset (i_* \mathcal{H})_0$  the local Gauss-Manin connection.

The local Gauss-Manin connection is a  $\mu$ -dimensional  $\mathbb{C}\{t\}[t^{-1}]$ -vectorspace and a regular  $\mathbb{C}\{t\}[\partial_t]$ -module. The generalized  $\alpha$ -eigenspaces  $C^{\alpha}$  of the operator  $t\partial_t$  define the decreasing filtration on  $G$  by free  $\mathbb{C}\{t\}$ -modules

$$V^{\alpha} = \bigoplus_{\alpha \leq \beta < \alpha+1} \mathbb{C}\{t\}C^{\beta}, \quad V^{>\alpha} = \bigoplus_{\alpha < \beta \leq \alpha+1} \mathbb{C}\{t\}C^{\beta}$$

of rank  $\mu$ , the  $V$ -filtration. In contrast to the  $\psi_\alpha$  and  $C^\alpha$ , the  $V^\alpha$  are independent of the coordinate  $t$ . The  $C^\alpha$  define a splitting

$$C^\alpha \cong V^\alpha / V^{>\alpha} = \text{gr}_V^\alpha G$$

of the  $V$ -filtration and we denote by  $\text{lead}_V$  the leading term with respect to this splitting. The ring  $\mathbb{C}\{t\}$  is a free module of rank 1 over the ring

$$\mathbb{C}\{\{s\}\} = \left\{ \sum_{k=0}^{\infty} a_k s^k \in \mathbb{C}[[s]] \mid \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathbb{C}\{t\} \right\}$$

where  $s = \int_0^1 dt$  acts by integration. This fact is generalized by the following lemma [13, 1.3.11].

**Lemma 4.** *The action of  $s = \partial_t^{-1}$  on  $V^{>-1}$  extends to a  $\mathbb{C}\{\{s\}\}$ -module structure and  $V^{>-1}$  is a free  $\mathbb{C}\{\{s\}\}$ -module of rank  $\mu$ .*

Since  $[\partial_t, t] = 1$ ,  $[t, s] = s^2$  and hence

$$t = s^2 \partial_s, \quad \partial_t t = s \partial_s.$$

We call a free  $\mathbb{C}\{\{s\}\}$ -submodule of  $V^{>-1}$  of rank  $\mu$  a  $\mathbb{C}\{\{s\}\}$ -lattice and call a  $t\partial_t$ -invariant  $\mathbb{C}\{\{s\}\}$ -lattice saturated. A basis  $\underline{e}$  of a  $\mathbb{C}\{\{s\}\}$ -lattice defines a matrix  $A = \sum_{k \geq 0} A_k s^k$  of  $t$  by  $t\underline{e} = \underline{e}A$  such that

$$t \cong A + s^2 \partial_s$$

is the basis representation of  $t$ .

#### 4. THE BRIESKORN LATTICE

The description of cohomology in terms of holomorphic differential forms by the de Rham isomorphism leads to the definition of the Brieskorn lattice

$$H'' = \Omega_{X,0}^{n+1} / df \wedge d\Omega_{X,0}^{n-1}.$$

By E. Brieskorn [2, 1.5] and M. Sebastiani [15], the Brieskorn lattice is the stalk at 0 of a locally free  $\mathcal{O}_T$ -sheaf  $\mathcal{H}''$  of rank  $\mu$  with  $\mathcal{H}''|_{T'} \cong \mathcal{H}$  and hence  $H'' \subset (i_* \mathcal{H})_0$ . The regularity of the Gauss-Manin connection proved by E. Brieskorn [2, 2.2] implies that  $H'' \subset G$ . B. Malgrange [8, 4.5] improved this result by the following theorem.

**Theorem 5.**  $H'' \subset V^{-1}$ .

By E. Brieskorn [2, 1.5], the Leray residue formula can be used to express the action of  $\partial_t$  in terms of differential forms by  $\partial_t[df \wedge \omega] = [d\omega]$ . In particular,  $sH'' \subset H''$  and

$$H'' / sH'' \cong \Omega_{X,0}^{n+1} / df \wedge \Omega_{X,0}^n \cong \mathbb{C}\{\underline{x}\} / \langle \underline{\partial}(f) \rangle.$$

Since the  $V^{>-1}$  is a  $\mathbb{C}\{\{s\}\}$ -module, theorem 5 implies that  $H''$  is a free  $\mathbb{C}\{\{s\}\}$ -module of rank  $\mu$  and the action of  $s$  can be expressed in terms of differential forms by

$$s[d\omega] = [df \wedge \omega].$$

For computational purposes, we may restrict our attention to the completion of the Brieskorn lattice. E. Brieskorn [2, 3.4] proved the following theorem.

**Theorem 6.** *The  $\mathfrak{m}_{X,0}$ - and  $\mathfrak{m}_{T,0}$ -adic topologies on  $H''$  coincide.*

While the proof of theorem 6 is highly non-trivial, the analogous statement for the  $\mathbb{C}\{\{s\}\}$ -structure of the Brieskorn lattice is quite elementary [13, 1.5.4].

**Proposition 7.** *The  $\mathfrak{m}_{X,0}$ - and  $\mathfrak{m}_{\mathbb{C}\{\{s\}\}}$ -adic topologies on  $H''$  coincide.*

We call the completion  $\widehat{H}''$  of  $H''$  the formal Brieskorn lattice. Since completion is faithfully flat,  $\widehat{H}''$  is a free  $\mathbb{C}[[s]]$ -module of rank  $\mu$  with a differential operator  $t = s^2\partial_s$ . The equality  $[\partial(f)\overline{g}d\underline{x}] = s[\partial(\overline{g})d\underline{x}]$  motivates to consider the differential relation  $\underline{\partial}(f) - s\underline{\partial}$ . It is not difficult to prove that it defines the formal Brieskorn lattice as a quotient of  $\mathbb{C}[[s, \underline{x}]]$  [13, 1.5.6].

**Proposition 8.**

$$\mathbb{C}[[s, \underline{x}]] \xrightarrow{\pi_H} \mathbb{C}[[s, \underline{x}]] / \langle \underline{\partial}(f) - s\underline{\partial} \rangle \mathbb{C}[[s, \underline{x}]] \cong_{\mathbb{C}[[s]]} \widehat{H}''.$$

Proposition 8 is the starting point for an algorithmic approach to the local Gauss-Manin connection. Let  $<_{\underline{x}}$  be a local degree ordering on  $\mathbb{C}[[\underline{x}]]$  such that  $\deg(\underline{x}) < \underline{0}$  and  $\deg(\underline{\partial}) = -\deg(\underline{x}) > \underline{0}$ . One can compute a polynomial standard basis  $\underline{g}$  of the Jacobian ideal  $\langle \underline{\partial}(f) \rangle$  and a polynomial transformation matrix  $B = (\overline{b}^j)^j$  such that  $\underline{g} = \underline{\partial}(f)B$ . By Nakayama's lemma,  $\underline{m} = (\underline{x}^{\underline{\beta}})_{\underline{x}^{\underline{\beta}} \notin \langle \text{lead}(\underline{g}) \rangle}$  represents a  $\mathbb{C}[[s]]$ -basis  $[\underline{m}]$  of  $\widehat{H}''$ . Let  $<_s$  be the local degree ordering on  $\mathbb{C}[[s]]$  and let  $< = (<_s, <_{\underline{x}})$  be the block ordering of  $<_s$  and  $<_{\underline{x}}$  on  $\mathbb{C}[[s, \underline{x}]]$ .

**Definition 9.**

- (1)  $\underline{h} = ((g_j - s\underline{\partial}\overline{b}^j)\underline{x}^{\underline{\beta}})_{j,\underline{\beta}}$ .
- (2)  $\deg(s) = \min \deg(\underline{m}) + 2 \min \deg(\underline{x}) < 0$ .
- (3)  $N = (N_K)_{K \geq 0}$  with  $N_K = K \deg(s) - 2 \min \deg(\underline{x})$ .
- (4)  $V = (V_K)_{K \geq 0}$  with  $V_K = \{p \in \mathbb{C}[[s, \underline{x}]] \mid \deg(p) < N_K\} + \langle s \rangle^K \subset \mathbb{C}[[s, \underline{x}]]$ .

Since  $\widehat{H}''$  is a free  $\mathbb{C}[[s]]$ -module,  $\underline{h}$  is a standard basis of the  $\mathbb{C}[[s]]$ -module  $\langle \underline{\partial}(f) - s\underline{\partial} \rangle \mathbb{C}[[s, \underline{x}]]$ . The following lemma is technical but not very deep and can be generalized to formal differential deformations [13, 2.2.10].

**Lemma 10.**  *$V = (V_K)_{K \geq 0}$  is a basis of the  $\langle s, \underline{x} \rangle$ -adic topology of  $\mathbb{C}[[s, \underline{x}]]$  with  $\pi_H(V_K) = \langle s \rangle^K \widehat{H}''$ . If  $s^\alpha \text{lead}(h_{j,\underline{\beta}}) \in V_K$  then  $s^\alpha h_{j,\underline{\beta}} \in V_K$ .*

Lemma 10 leads to a normal form algorithm for the Brieskorn lattice [13, 2.2.12]. It computes a normal form with respect to  $\underline{h}$  and hence

the  $[\underline{m}]$ -basis representation in  $H''$ . The normal form computation up to a given degree can be continued up to any higher degree without additional computational effort. The normal form algorithm for the Brieskorn lattice is a special case of a modification of Buchberger's normal form algorithm [3] for power series rings where termination is replaced by adic convergence [13, 2.1.19].

## 5. MIXED HODGE STRUCTURE

By lemma 2, there is a  $\mathbb{C}$ -isomorphism

$$H_{\mathbb{C}} = \bigoplus_{-1 < \alpha \leq 0} H_{\mathbb{C}}^{e^{-2\pi i \alpha}} \xrightarrow{\psi} \bigoplus_{-1 < \alpha \leq 0} C^{\alpha} \cong V^{>-1}/sV^{>-1}$$

defined by  $\psi = \bigoplus_{-1 < \alpha \leq 0} \psi_{\alpha}$  and the monodromy  $M$  on  $H_{\mathbb{C}}$  corresponds to  $\exp(-2\pi i t \partial_t)$  on  $\bigoplus_{-1 < \alpha \leq 0} C^{\alpha}$ .

The Hodge filtration  $F = (F_k)_{k \in \mathbb{Z}}$  on  $V^{>-1}$  defined by J. Scherk and J.H.M. Steenbrink [14] is the increasing filtration by the free  $\mathbb{C}\{\{s\}\}$ -modules

$$F_k = F^{n-k} = (s^{-k} H'') \cap V^{>-1}$$

of rank  $\mu$ . Via the splitting  $C^{\alpha} \cong \text{gr}_V^{\alpha} V^{>-1}$ , the Hodge filtration induces an increasing Hodge filtration  $FC^{\alpha}$  by  $\mathbb{C}$ -vectorspaces on  $C^{\alpha}$  and, via  $\psi$ , on  $H_{\mathbb{C}}$ . The nilpotent operator  $-2\pi i N \in \text{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$  defines an increasing weight filtration  $W = (W_k)_{k \in \mathbb{Z}}$  centered at  $n$  resp.  $n+1$  on  $H_{\mathbb{Q}}^{\neq 1}$  resp.  $H_{\mathbb{Q}}^1$ .

**Theorem 11.** *The weight filtration  $W$  on  $H_{\mathbb{Q}}$  and the Hodge filtration  $F$  on  $H_{\mathbb{C}}$  define a mixed Hodge structure on the cohomology  $H$  of the Milnor fibre and the operator  $N$  is a morphism of mixed Hodge structures of type  $(-1, -1)$ .*

The mixed Hodge structure on the cohomology of the Milnor fibre was discovered by J.H.M. Steenbrink [16] and described in terms of the Brieskorn lattice by A.N. Varchenko [17].

The nilpotent operator  $N$  on  $C^{\alpha}$  defines an increasing weight filtration  $W = (W_k)_{k \in \mathbb{Z}}$  centered at  $n$  on  $C^{\alpha}$ . By definition  $N$  commutes with  $\psi_{\alpha}$  and hence

$$\psi_{\alpha}(W H_{\mathbb{C}}^{e^{-2\pi i \alpha}}) = \begin{cases} W C^{\alpha}, & \alpha \notin \mathbb{Z}, \\ W[-1] C^{\alpha}, & \alpha \in \mathbb{Z}. \end{cases}$$

The weight filtration  $W = \bigoplus_{-1 < \alpha \leq 0} \mathbb{C}\{\{s\}\} W C^{\alpha}$  on  $V^{>-1}$  by free  $\mathbb{C}\{\{s\}\}$ -modules induces  $W C^{\alpha}$  via the splitting  $C^{\alpha} \cong \text{gr}_V^{\alpha} V^{>-1}$ .

The spectral pairs are those pairs  $(\alpha, l) \in \mathbb{Q} \times \mathbb{Z}$  with positive multiplicity

$$d_l^{\alpha} = \dim_{\mathbb{C}} \text{gr}_l^W \text{gr}_V^{\alpha} \text{gr}_0^F V^{>-1}.$$

Via the isomorphism  $\psi$ , they correspond to the Hodge numbers

$$h_{\lambda}^{p, l-p} = \dim_{\mathbb{C}} \text{gr}_F^p \text{gr}_l^W H_{\mathbb{C}}^{\lambda}$$

by  $d_l^{\alpha+p} = h_{e^{-2\pi i\alpha}}^{n-p, l-n+p}$  for  $-1 < \alpha < 0$  and  $d_l^p = h_1^{n-p, l+1-n+p}$  and inherit the symmetry properties

$$d_l^\alpha = d_l^{2n-l-1-\alpha}, \quad d_l^\alpha = d_{2n-l}^{\alpha-n+l}, \quad d_l^\alpha = d_{2n-l}^{n-1-\alpha}$$

from the mixed Hodge structure. The spectral numbers are those numbers  $\alpha \in \mathbb{Q}$  with positive multiplicity

$$d^\alpha = \dim_{\mathbb{C}} \operatorname{gr}_V^\alpha \operatorname{gr}_0^F V^{>-1} = \sum_{l \in \mathbb{Z}} d_l^\alpha$$

and have the symmetry property  $d^\alpha = d^{n-1-\alpha}$ .

## 6. M. SAITO'S BASIS

By P. Deligne [5, 1.2.8], a morphism of mixed Hodge structures is strict for the Hodge filtration. In particular, by theorem 11,  $N$  is strict for the Hodge filtration on  $H_{\mathbb{C}}$  and on  $\operatorname{gr}_V V^{>-1}$ . Hence, there is a direct sum decomposition  $F_k C^\alpha = \bigoplus_{j \leq k} C^{\alpha, j}$  such that  $N(C^{\alpha, k}) \subset C^{\alpha, k+1}$ , and  $sC^{\alpha, k} \subset C^{\alpha+1, k-1}$ . By definition of the Hodge filtration,

$$\operatorname{lead}_V(H'') = \sum_{\alpha \in \mathbb{Q}} \sum_{k \leq 0} \mathbb{C}\{\{s\}\} C^{\alpha, k} = \bigoplus_{\alpha \in \mathbb{Q}} \mathbb{C}\{\{s\}\} G^\alpha$$

where  $G^\alpha = C^{\alpha, 0}$ . Let  $<_{\mathbb{Q} \times \mathbb{Z}} = (>_{\mathbb{Q}}, >_{\mathbb{Z}})$  be the block ordering of  $>_{\mathbb{Q}}$  and  $>_{\mathbb{Z}}$  on the index set  $\mathbb{Q} \times \mathbb{Z}$ . Then the Hodge filtration defines a refinement of the  $V$ -filtration on  $V^{>-1}$  by free  $\mathbb{C}\{\{s\}\}$ -modules  $V^{\alpha, k} = F_k C^\alpha \oplus V^{>\alpha}$  of rank  $\mu$  and the  $C^{\alpha, k}$  define a splitting of this refined filtration compatible with  $s$ . We call the refinement the Hodge refinement and the splitting a Hodge splitting. The following lemma follows essentially from the fact that  $\mathbb{C}\{\{s\}\}$  is a discrete valuation ring [13, 1.10.5, 1.10.10].

**Lemma 12.** *Let  $H$  be a  $\mathbb{C}\{\{s\}\}$ -lattice and  $C^{\alpha, k}$  a splitting of a refinement of the  $V$ -filtration compatible with  $s$ . Then a minimal standard basis of  $H$  is a  $\mathbb{C}\{\{s\}\}$ -basis and there is a reduced minimal standard basis of  $H$ .*

In particular, there is a reduced minimal standard basis of  $H''$  for a Hodge splitting. The following proposition follows essentially from lemma 2.3 [13, 1.10.12].

**Proposition 13.** *Let  $\underline{h}$  be a reduced minimal standard basis of  $H''$  for a Hodge splitting. Then the  $\underline{h}$ -matrix  $A$  of  $t$  has degree 1. In particular,*

$$(H'', t) \xleftarrow{\underline{h}} (\mathbb{C}\{\{s\}\}^\mu, A_0 + A_1 s + s^2 \partial_s)$$

*is an isomorphism. Moreover,  $A_1$  is semisimple with eigenvalues the spectral numbers of  $f$  added by 1 and  $\operatorname{gr}_V(A_0)$  can be identified with  $N$ .*

Note that the matrices  $A_0$  and  $A_1$  in proposition 13 determine the differential structure of the Brieskorn lattice. M. Saito [10] first constructed a  $\mathbb{C}\{\{s\}\}$ -basis of  $H''$  as in proposition 13 without calling it a reduced minimal standard basis.

## 7. THE ALGORITHM

We describe an algorithm to compute  $A_0$  and  $A_1$  as in proposition 13 [13]. This algorithm can be simplified to compute the complex monodromy, the spectral numbers, or the spectral pairs only [13].

The normal form algorithm for the Brieskorn lattice in section 4 computes the  $[\underline{m}]$ -matrix  $A = \sum_{k \geq 0} A_k s^k$  of  $t$  defined by  $t[\underline{m}] = [f\underline{m}] = [\underline{m}]A$  up to any degree. We identify the columns of a matrix  $H$  with the generators of a submodule  $\langle H \rangle \subset \mathbb{C}[[s]]^\mu$  and denote by  $E$  the unit matrix. Then  $\langle E \rangle$  is the  $[\underline{m}]$ -basis representation of  $\widehat{H}''$ . Hence, the following two statements hold for  $\underline{h} = [\underline{m}]$  with  $\kappa = 0$  and  $H = E$ .

( $H_{\underline{h}}$ ) One can compute  $\kappa \geq 0$  and a  $\mu \times \mu$ -matrix  $H$  with coefficients in  $\mathbb{C}[s]$  of degree at most  $\kappa$  such that  $\langle H \rangle$  is the  $\underline{h}$ -basis representation of  $\widehat{H}''$  and  $s^\kappa \langle E \rangle \subset \langle H \rangle$ .

( $A_{\underline{h}}$ ) One can compute the  $\underline{h}$ -matrix  $A$  of  $t$  up to any degree.

Step by step, we improve the  $\mathbb{C}[[s]]$ -basis  $\underline{h}$  and show that ( $H_{\underline{h}}$ ) and ( $A_{\underline{h}}$ ) hold. After the last step,  $A_0$  and  $A_1$  as in proposition 13 can be computed by a basis transformation of  $A$  to a reduced minimal standard basis of  $\langle H \rangle$  up to a certain degree bound.

We call the canonical projection  $\text{jet}_k : \mathbb{C}[[s]] \rightarrow \bigoplus_{j=0}^k \mathbb{C}s^j$  the  $k$ -jet. Let the monomial ordering on  $\mathbb{C}[[s]]^\mu = \mathbb{C}[[s]] \otimes_{\mathbb{C}} \mathbb{C}^\mu$  be the block ordering  $< = (<_s, >_\mu)$  of the local degree ordering  $<_s$  on  $\mathbb{C}[[s]]$  and the inverse ordering  $>_\mu$  on the indices of the basis elements of  $\mathbb{C}^\mu$ .

**7.1. The Saturation of  $H''$ .** In this step, we show that ( $H_{\underline{h}}$ ) and ( $A_{\underline{h}}$ ) hold for a  $\mathbb{C}[[s]]$ -basis  $\underline{h}$  of a saturated  $\mathbb{C}[[s]]$ -lattice.

The increasing sequence of  $\mathbb{C}[[s]]$ -lattices defined by

$$\widehat{H}_0'' = \widehat{H}'', \quad \widehat{H}_{k+1}'' = s\widehat{H}_k'' + t\widehat{H}_k'' \subset \widehat{H}''$$

is stationary since  $\widehat{H}''$  is noetherian. Hence, the saturation  $\widehat{H}_\infty'' = \bigcup_{k \geq 0} \widehat{H}_k''$  of  $\widehat{H}''$  is a saturated  $\mathbb{C}[[s]]$ -lattice. The  $[\underline{m}]$ -basis representation  $\langle H_k \rangle$  of  $\widehat{H}_k''$  can be computed by

$$H_0 = Q_{-1} = E, \quad Q_k = (\text{jet}_k(A) + s^2 \partial_s) Q_{k-1}, \quad H_{k+1} = (sH_k | Q_k).$$

We successively compute the  $H_k$  and check in each step if  $\langle Q_k \rangle \subset \langle H_k \rangle$  by a standard basis and normal form computation. If  $\langle Q_k \rangle \subset \langle H_k \rangle$  then we stop the computation and set  $\kappa = k$  and  $H_\infty = H_\kappa$ . Then  $\langle H_\infty \rangle$  is the  $[\underline{m}]$ -basis representation of  $\widehat{H}_\infty''$ . We replace  $H_\infty$  by a minimal standard basis of  $\langle H_\infty \rangle$ . Then  $\underline{h} = s^{-\kappa} \underline{h} H_\infty$  is a  $\mathbb{C}[[s]]$ -basis of a saturated  $\mathbb{C}[[s]]$ -lattice. By a normal form computation with respect to

$H_\infty$  up to degree  $\kappa$ , we compute the  $\underline{h}$ -basis representation  $\langle H_\infty^{-1} s^\kappa E \rangle = \langle \text{jet}_\kappa(H_\infty^{-1} s^\kappa E) \rangle$  of  $\widehat{H}''$ . Since  $\langle H_\infty \rangle \subset \langle E \rangle$ ,  $s^\kappa \langle E \rangle \subset \langle H_\infty^{-1} s^\kappa E \rangle$ . By a normal form computation with respect to  $H_\infty$  up to degree  $\kappa + k$ , one can compute the  $k$ -jet

$$\text{jet}_k(H_\infty^{-1}(A - \kappa s E + s^2 \partial_s) H_\infty) = \text{jet}_k(H_\infty^{-1}(\text{jet}_{\kappa+k}(A - \kappa s E) + s^2 \partial_s) H_\infty)$$

of the  $\underline{h}$ -matrix of  $t$  for any  $k \geq 0$ .

**7.2. The V-Filtration.** In this step, we show that  $(H_{\underline{v}})$  and  $(A_{\underline{v}})$  hold for a  $<_{\mathbb{Q}}$ -increasingly ordered  $\mathbb{C}[[s]]$ -basis  $\underline{v}$  of a  $\widehat{V}^\alpha$  compatible with the direct sum decomposition  $\widehat{V}^\alpha / s \widehat{V}^\alpha \cong \bigoplus_{\alpha \leq \beta < \alpha+1} C^\beta$ .

Since  $\underline{h}$  is a  $\mathbb{C}[[s]]$ -basis of a saturated  $\mathbb{C}\{\{s\}\}$ -lattice,  $A_0 = 0$  and, by theorem 1, the eigenvalues of  $A_1$  are rational. In order to compute the eigenvalues of  $A_1$ , we transform  $A_1$  to Hessenberg form and factorize the characteristic polynomials of its blocks. Then we compute a constant  $\mathbb{C}[[s]]$ -basis transformation such that  $A_1 = \text{diag}(\alpha_1, \dots, \alpha_\mu) + N$  with  $\alpha_1 \leq \dots \leq \alpha_\mu$  where  $\text{diag}(\alpha_1, \dots, \alpha_\mu)$  denotes the diagonal matrix with entries  $\alpha_1, \dots, \alpha_\mu$ . If  $\alpha_\mu - \alpha_1 < 1$  then  $\underline{v} = \underline{h}$  is a  $<_{\mathbb{Q}}$ -increasingly ordered  $\mathbb{C}[[s]]$ -basis  $\underline{v}$  of a  $\widehat{V}^\alpha$  compatible with the direct sum decomposition  $\widehat{V}^\alpha / s \widehat{V}^\alpha \cong \bigoplus_{\alpha \leq \beta < \alpha+1} C^\beta$ . If  $\alpha_\mu - \alpha_1 \geq 1$  then we proceed as follows. Let

$$A = \begin{pmatrix} A^{1,1} & A^{1,2} \\ A^{2,1} & A^{2,2} \end{pmatrix}$$

such that  $A_0 = 0$ ,  $A_1^{1,2} = 0$ ,  $A_1^{2,1} = 0$ , and the eigenvalues of  $A_1^{1,1}$  are the eigenvalues  $\alpha$  of  $A_1$  with  $\alpha < \alpha_1 + 1$ . Then the  $\mathbb{C}[[s]][s^{-1}]$ -basis transformation

$$H \mapsto \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 1 \end{pmatrix} H, \quad A \mapsto \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 1 \end{pmatrix} (A + s^2 \partial_s) \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A^{1,1} + s & \frac{1}{s} A^{1,2} \\ s A^{2,1} & A^{2,2} \end{pmatrix}$$

decreases  $\alpha_\mu - \alpha_1$  and the degree up to which  $A$  is computed by 1 and increases  $\kappa$  by 1. After at most  $n$  such transformations,  $\alpha_\mu - \alpha_1 < 1$ .

**7.3. The Canonical V-Splitting.** In this step, we show that  $(H_{\underline{c}})$  and  $(A_{\underline{c}})$  hold for a  $<_{\mathbb{Q}}$ -increasingly ordered  $\mathbb{C}$ -basis  $\underline{c}$  of a direct sum  $\bigoplus_{\alpha \leq \beta < \alpha+1} C^\beta$  compatible with the direct sum.

Let  $\underline{c}$  be the image of  $[\underline{v}]$  under the splitting  $\widehat{V}^\alpha / s \widehat{V}^\alpha \cong \bigoplus_{\alpha \leq \beta < \alpha+1} C^\beta$ . By Nakayama's lemma,  $\underline{c}$  is a  $\mathbb{C}$ -basis of  $\bigoplus_{\alpha \leq \beta < \alpha+1} C^\beta$  compatible with the direct sum. The eigenvalues of the commutator  $[\cdot, A_1] \in \text{End}_{\mathbb{C}}(\mathbb{C}^{\mu^2})$  are the differences of the eigenvalues of  $A_1$ . Since  $\alpha_\mu - \alpha_1 < 1$ ,  $[\cdot, A_1] - k \in \text{GL}_{\mu^2}(\mathbb{C})$  for  $k \geq 1$ . Let  $U = \sum_{j=0}^{\infty} U_j s^j$  be the  $\mathbb{C}[[s]]$ -basis transformation defined by  $\underline{c} = \underline{v} U$ . Then  $U_0 = E$  and  $U A_1 s = (A + s^2 \partial_s) U$  or equivalently

$$U_k = ([\cdot, A_1] - k)^{-1} \sum_{j=0}^{k-1} A_{k-j+1} U_j$$



for  $k \geq 1$  and hence one can compute  $U$  up to any degree. Since  $U_0 = E$  and  $\kappa \geq 0$ ,  $\text{jet}_\kappa(U)$  is a minimal standard basis of  $\langle E \rangle$ . By a normal form computation with respect to  $U$  up to degree  $\kappa$ , we compute the  $\underline{c}$ -basis representation  $\langle U^{-1}H \rangle = \langle \text{jet}_\kappa(\text{jet}_\kappa(U)^{-1}H) \rangle$  of  $\widehat{H}''$  and  $A_1 s$  is the  $\underline{c}$ -matrix of  $t$ .

#### 7.4. A Hodge Splitting.

In this step, we show that  $(H_{\underline{f}})$  and  $(A_{\underline{f}})$  hold for a  $<_{\mathbb{Q} \times \mathbb{Z}}$ -decreasingly ordered  $\mathbb{C}$ -basis  $\underline{f}$  of a direct sum  $\bigoplus_{\alpha \leq \beta < \alpha+1} \bigoplus_{k \in \mathbb{Z}} C^{\beta,k}$  compatible with the direct sum and that one can compute  $A_0$  and  $A_1$  as in proposition 13.

We compute a standard basis of  $H$  up to degree  $\kappa$  in order to compute the  $\underline{c}$ -basis representation of the Hodge filtration  $F$ . The nilpotent part of  $A_1$  is the  $\underline{c}$ -basis representation of  $N$ . By computing images and quotients of  $\mathbb{C}$ -vectorspaces, we compute the  $\underline{c}$ -basis representation of a Hodge splitting  $F_k C^\beta = \bigoplus_{j \leq k} C^{\beta,j}$ . Then we compute a constant  $\mathbb{C}[[s]]$ -basis transformation  $\underline{f} = \underline{c}\underline{U}$  such that  $\underline{f}$  is a  $<_{\mathbb{Q} \times \mathbb{Z}}$ -decreasingly ordered  $\mathbb{C}$ -basis of the direct sum  $\bigoplus_{\alpha \leq \beta < \alpha+1} \bigoplus_{k \in \mathbb{Z}} C^{\beta,k}$  compatible with the direct sum.

We replace  $H$  by a reduced minimal standard basis of  $\langle H \rangle$  up to degree  $\kappa + 1$ . By a normal form computation with respect to  $H$  up to degree  $\kappa + 1$ , we compute the 1-jet

$$\text{jet}_1(H^{-1}(A + s^2 \partial_s)H) = \text{jet}_1(\text{jet}_{\kappa+1}(H)^{-1}(\text{jet}_{\kappa+1}(A) + s^2 \partial_s)\text{jet}_{\kappa+1}(H))$$

of the  $\underline{c}H$ -matrix  $A$  of  $t$  in order to compute  $A_0$  and  $A_1$  as in proposition 10.

## 8. AN EXAMPLE

The algorithm in section 7 is implemented in the computer algebra system SINGULAR [6] in the procedure `tmatrix` in the library `gaussman.lib` [12]. In an example SINGULAR session, we compute the differential structure of the Brieskorn lattice of the singularity of type  $T_{2,5,5}$  defined by the polynomial  $f = x^2 y^2 + x^5 + y^5$ .

First, we load the SINGULAR library `gaussman.lib`:

```
> LIB "gaussman.lib";
```

Then, we define the local ring  $R = \mathbb{Q}[x, y]_{\langle x, y \rangle}$  with the local degree ordering `ds` as monomial ordering and the polynomial  $f = x^2 y^2 + x^5 + y^5 \in R$ :

```
> ring R=0,(x,y),ds;
> poly f=x2y2+x5+y5;
```

Finally, we compute  $A_0$  and  $A_1$  as in proposition 10:

```
> list A=tmatrix(f);
```

The result is the list  $\mathbf{A}=\mathbf{A}[1], \mathbf{A}[2]$  such that  $\mathbf{A}[i+1] = A_i$  and

$$A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad A_1 = \text{diag}\left(\frac{1}{2}, \frac{7}{10}, \frac{7}{10}, \frac{9}{10}, \frac{9}{10}, 1, \frac{11}{10}, \frac{11}{10}, \frac{13}{10}, \frac{13}{10}, \frac{3}{2}\right).$$

By proposition 10,  $(H'', t) \cong (\mathbb{C}\{\{s\}\}^\mu, A_0 + sA_1 + s^2\partial_s)$  and the spectral pairs are  $(-\frac{1}{2}, 2)$ ,  $(-\frac{3}{10}, 1)^2$ ,  $(-\frac{1}{10}, 1)^2$ ,  $(0, 1)$ ,  $(\frac{1}{10}, 1)^2$ ,  $(\frac{3}{10}, 1)^2$ ,  $(\frac{1}{2}, 0)$ .

## REFERENCES

- [1] V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko. *Singularities of Differentiable Maps*, volume II. Birkhäuser, 1988.
- [2] E. Brieskorn. Die Monodromie der isolierten Singularitäten von Hyperflächen. *Manuscr. Math.*, 2:103–161, 1970.
- [3] B. Buchberger. *Gröbner bases: an algorithmic method in polynomial ideal theory*. In N.K. Bose, editor, *Recent trends in multidimensional system theory*. Reidel, 1985.
- [4] P. Deligne. *Equations différentielles à points singuliers réguliers*, volume 163 of *Lect. Notes Math.* Springer, 1970.
- [5] P. Deligne. Théorie de Hodge, II. *Publ. Math. I.H.E.S.*, 40:5–57, 1972.
- [6] G.-M. Greuel, G. Pfister, and H. Schönemann. SINGULAR 2.0.3. A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2002. <http://www.singular.uni-kl.de>.
- [7] E.J.N. Looijenga. *Isolated Singular Points on Complete Intersections*, volume 77 of *LMS Lect. Note Series*. Cambr. Univ. Press, 1984.
- [8] B. Malgrange. Intégrales asymptotiques et monodromie. *Ann. scient. Ec. Norm. Sup.*, 7:405–430, 1974.
- [9] J. Milnor. *Singular Points on Complex Hypersurfaces*, volume 61 of *Ann. Math. Stud.* Princ. Univ. Press, 1968.
- [10] M. Saito. On the structure of Brieskorn lattices. *Ann. Inst. Fourier Grenoble*, 39:27–72, 1989.
- [11] M. Schulze. Algorithms for the Gauss-Manin connection. *Journ. Symb. Comp.*, 32(5):549–564, 2001.
- [12] M. Schulze. `gaussman.lib`. SINGULAR 2.0.3 library, Centre for Computer Algebra, University of Kaiserslautern, 2001. <http://www.singular.uni-kl.de>.
- [13] M. Schulze. *Algorithmic Gauss-Manin Connection*. PhD thesis, University of Kaiserslautern, 2002.
- [14] J. Scherk and J.H.M. Steenbrink. On the mixed Hodge structure on the cohomology of the Milnor fibre. *Math. Ann.*, 271:641–655, 1985.
- [15] M. Sebastiani. Preuve d’une conjecture de Brieskorn. *Manuscr. Math.*, 2:301–308, 1970.
- [16] J. Steenbrink. Mixed Hodge structure on the vanishing cohomology. In *Real and complex singularities*, pages 525–562. Nordic summer school, Oslo, 1976.
- [17] A.N. Varchenko. Asymptotic Hodge structure in the vanishing cohomology. *Math. USSR Izvestija*, 18(3):496–512, 1982.

M. SCHULZE, DEPARTMENT OF MATHEMATICS, D-67653 KAISERSLAUTERN  
*E-mail address:* mschulze@mathematik.uni-kl.de